Interval analysis with application to the safe navigation of autonomous vehicles

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Interval analysis
Problem. Given \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) and a box \([x] \subset \mathbb{R}^n\), prove that \( \forall x \in [x], f(x) \geq 0 \).

Interval arithmetic can solve efficiently this problem.
Example. Is the function

\[ f(x) = x_1 x_2 - (x_1 + x_2) \cos x_2 + \sin x_1 \cdot \sin x_2 + 2 \]

always positive for \( x_1, x_2 \in [-1, 1] \)?
Interval arithmetic

\[
[-1, 3] + [2, 5] = ?, \\
[-1, 3] \cdot [2, 5] = ?, \\
\text{abs}([-7, 1]) = ?
\]
Interval arithmetic

\([-1, 3] + [2, 5] = [1, 8],\)
\([-1, 3] \cdot [2, 5] = [-5, 15],\)
\(\text{abs}([-7, 1]) = [0, 7]\)
The interval extension of

\[ f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2 \]

is

\[ [f]([x_1], [x_2]) = [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos[x_2] + \sin[x_1] \cdot \sin[x_2] + 2. \]
**Theorem** (Moore, 1970)

\[
[f](\mathbf{x}) \subset \mathbb{R}^+ \Rightarrow \forall \mathbf{x} \in \mathbf{x}, f(\mathbf{x}) \geq 0.
\]
Interval arithmetic
If $\diamond \in \{+, -, \cdot, /, \max, \min\}$

$$[x] \diamond [y] = [\{x \diamond y | x \in [x], y \in [y]\}].$$

where $[A]$ is the smallest interval which encloses $A \subset \mathbb{R}$. 
Exercise.

\[ [-1, 3] + [2, 5] = [?, ?] \]
\[ [-1, 3] \cdot [2, 5] = [?, ?] \]
\[ [-2, 6] / [2, 5] = [?, ?] \]
Solution.

\[ [-1,3] + [2,5] = [1,8] \]
\[ [-1,3] \cdot [2,5] = [-5,15] \]
\[ [-2,6] / [2,5] = [-1,3] \]
Exercise. Compute

\[-2, 2] / [-1, 1] = [?, ?]
Solution.

$$[-2, 2]/[-1, 1] = [-\infty, \infty]$$
\[
[x^-, x^+] + [y^-, y^+] = [x^- + y^-, x^+ + y^+], \\
[x^-, x^+] \cdot [y^-, y^+] = [x^- y^- \land x^+ y^- \land x^- y^+ \land x^+ y^+, \\
x^- y^- \lor x^+ y^- \lor x^- y^+ \lor x^+ y^+],
\]
If \( f \in \{ \cos, \sin, \text{sqr}, \sqrt{\cdot}, \log, \exp, \ldots \} \)

\[
f([x]) = \left[ \{ f(x) \mid x \in [x] \} \right].
\]
Exercise.

\[
\begin{align*}
\sin([0, \pi]) &= \,? \\
\text{sqr}([-1, 3]) &= [-1, 3]^2 = ? \\
\text{abs}([-7, 1]) &= \,? \\
\sqrt{[-10, 4]} &= \sqrt{[-10, 4]} = ? \\
\log([-2, -1]) &= \,?.
\end{align*}
\]
Solution.

\[
\begin{align*}
\sin([0, \pi]) &= [0, 1] \\
sqr([-1, 3]) &= [-1, 3]^2 = [0, 9] \\
abs([-7, 1]) &= [0, 7] \\
sqrt([-10, 4]) &= \sqrt{-10, 4} = [0, 2] \\
log([-2, -1]) &= \emptyset.
\end{align*}
\]
Inclusion functions
A box, or interval vector $[x]$ of $\mathbb{R}^n$ is

$$[x] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$. 
\[ \mathbf{f} : \mathbb{I}R^n \rightarrow \mathbb{I}R^m \] is an *inclusion function* for \( \mathbf{f} \) if

\[ \forall \mathbf{x} \in \mathbb{I}R^n, \quad \mathbf{f}(\mathbf{x}) \subset \mathbf{f}^*(\mathbf{x}). \]

Inclusion functions \( \mathbf{f} \) and \( \mathbf{f}^* \); here, \( \mathbf{f}^* \) is minimal.
The inclusion function $[f]$ is

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>monotonic</td>
<td>if $([x] \subset [y]) \Rightarrow ([f([x]) \subset [f([y])])$</td>
</tr>
<tr>
<td>minimal</td>
<td>if $\forall [x] \in \mathbb{R}^n, [f([x]) = [f([x])]$</td>
</tr>
<tr>
<td>thin</td>
<td>if $w([x]) = 0 \Rightarrow w([f([x]) = 0$</td>
</tr>
</tbody>
</table>
| convergent     | if $w([x]) \to 0 \Rightarrow w([f([x]) \to 0$.
Interval analysis
Contractors
Tubes

Interval analysis with application to the safe navigation of autonomous vehicles
Exercise. The natural inclusion function for \( f(x) = x^2 + 2x + 4 \) is

\[
[f](\[x\]) = [x]^2 + 2[x] + 4.
\]

For \([x] = [-3, 4]\), compute \([f][x]\) and \(f([x])\).
Solution. If \([x] = [-3, 4]\), we have

\[
[f]([-3, 4]) = [-3, 4]^2 + 2[-3, 4] + 4
= [0, 16] + [-6, 8] + 4
= [-2, 28].
\]

Note that \(f([-3, 4]) = [3, 28] \subset [f]([-3, 4]) = [-2, 28]\).
A minimal inclusion function for

\[ f: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \]

\[ (x_1, x_2) \mapsto (x_1 x_2, x_1^2, x_1 - x_2). \]

is

\[ [f]: \mathbb{IR}^2 \rightarrow \mathbb{IR}^3 \]

\[ ([x_1], [x_2]) \mapsto ([x_1] \cdot [x_2], [x_1]^2, [x_1] - [x_2]). \]
If \( f \) is given by

\[
\begin{align*}
\text{Algorithm } f(&\text{in : } x = (x_1, x_2, x_3), \text{ out : } y = (y_1, y_2)) \\
z := x_1 \\
fork := 0 \text{ to } 100 \\
\quad z := x_2(z + k \cdot x_3) \\
\text{next} \\
y_1 := z \\
y_2 := \sin(zx_1)
\end{align*}
\]
Its natural inclusion function is

Algorithm \( \mathbf{f}(\text{in} : \mathbf{x} = ([x_1], [x_2], [x_3]), \text{out} : \mathbf{y} = ([y_1], [y_2])) \)

\[
[z] := [x_1] \\
\text{for } k := 0 \text{ to } 100 \\
\quad [z] := [x_2] \cdot ([z] + k \cdot [x_3]) \\
\text{next} \\
[y_1] := [z] \\
[y_2] := \sin([z] \cdot [x_1])
\]

Is \( \mathbf{f} \) convergent? thin? monotonic?
Set inversion
A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$. Compact sets $X$ can be bracketed between inner and outer subpavings:

$$X^- \subset X \subset X^+.$$
Example.

$$X = \{(x_1, x_2) \mid x_1^2 + x_2^2 \in [1, 2]\}.$$
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and let $Y$ be a subset of $\mathbb{R}^m$. Set inversion is the characterization of

$$X = \{ x \in \mathbb{R}^n \mid f(x) \in Y \} = f^{-1}(Y).$$
We shall use the following tests.

(i) \([f([x])] \subset Y \Rightarrow [x] \subset X\)

(ii) \([f([x])] \cap Y = \emptyset \Rightarrow [x] \cap X = \emptyset\).

Boxes for which these tests failed, will be bisected, except if they are too small.
Localization
A robot measures distances to three beacons.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x_i$</th>
<th>$y_i$</th>
<th>$[d_i]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>[1,2]</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>[2,3]</td>
</tr>
<tr>
<td>3</td>
<td>−1</td>
<td>−1</td>
<td>[3,4]</td>
</tr>
</tbody>
</table>

The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$. 
Define

\[ \mathcal{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} \in [d_i] \right\}. \]
\[
\begin{align*}
\mathbb{P}(p \in \mathcal{P}^0) &= 0.729 \\
\mathbb{P}(p \in \mathcal{P}^1) &= 0.972 \\
\mathbb{P}(p \in \mathcal{P}^2) &= 0.999
\end{align*}
\]
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Contractors
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To characterize $X \subset \mathbb{R}^n$, bisection algorithms bisect all boxes in all directions and become inefficient. Interval methods can still be useful if

- the solution set $X$ is small (optimization problem, solving equations),
- contraction procedures are used as much as possible,
- bisections are used only as a last resort.
The operator $\mathcal{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a \textit{contractor} for $X \subset \mathbb{R}^n$ if
\[
\forall [x] \in \mathbb{IR}^n, \left\{ \begin{array}{l}
\mathcal{C}([x]) \subset [x] \\
\mathcal{C}([x]) \cap X = [x] \cap X
\end{array} \right. \quad \text{(contractance),}
\]
\[
\quad \text{(completeness).}
\]
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Interval analysis with application to the safe navigation of autonomous vehicles
The operator $\mathcal{C} : \mathbb{IR}^n \rightarrow \mathbb{IR}^n$ is a *contractor* for the equation $f(x) = 0$, if

$$\forall [x] \in \mathbb{IR}^n, \left\{ \begin{array}{l} \mathcal{C}([x]) \subset [x] \\ x \in [x] \text{ et } f(x) = 0 \Rightarrow x \in \mathcal{C}([x]) \end{array} \right.$$
Exercice. Let \( x, y, z \) be 3 variables such that

\[
\begin{align*}
x & \in [\infty, 5], \\
y & \in [\infty, 4], \\
z & \in [6, \infty], \\
z & = x + y.
\end{align*}
\]

Contract the intervals for \( x, y, z \).
Solution.

\[ x = [2, 5] \]
\[ y = [1, 4] \]
\[ z = [6, 9] \]
Since $x \in [-\infty, 5], y \in [-\infty, 4], z \in [6, \infty]$ and $z = x + y$, we have

\[
z = x + y \Rightarrow z \in [6, \infty] \cap ([-\infty, 5] + [-\infty, 4]) = [6, \infty] \cap [-\infty, 9] = [6, 9].
\]

\[
x = z - y \Rightarrow x \in [-\infty, 5] \cap ([6, \infty] - [-\infty, 4]) = [-\infty, 5] \cap [2, \infty] = [2, 5].
\]

\[
y = z - x \Rightarrow y \in [-\infty, 4] \cap ([6, \infty] - [-\infty, 5]) = [-\infty, 4] \cap [1, \infty] = [1, 4].
\]
The contractor associated with $z = x + y$ is:

```
Algorithm pplus (inout: [z], [x], [y])

[z] := [z] ∩ ([x] + [y])        // z = x + y
[x] := [x] ∩ ([z] − [y])        // x = z − y
[y] := [y] ∩ ([z] − [x])        // y = z − x
```
The contractor associated with $z = x \cdot y$ is:

<table>
<thead>
<tr>
<th>Algorithm pmult (inout: $[z], [x], [y]$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[z] := [z] \cap ([x] \cdot [y])$</td>
</tr>
<tr>
<td>$[x] := [x] \cap ([z] \cdot 1/[y])$</td>
</tr>
<tr>
<td>$[y] := [y] \cap ([z] \cdot 1/[x])$</td>
</tr>
</tbody>
</table>

// $z = x \cdot y$
// $x = z/y$
// $y = z/x$
The contractor associated with $y = \exp x$ is:

<table>
<thead>
<tr>
<th>Algorithm pexp (inout: $[y], [x]$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 $[y] := [y] \cap \exp ([x])$</td>
</tr>
<tr>
<td>2 $[x] := [x] \cap \log ([y])$</td>
</tr>
</tbody>
</table>
Any constraint for which such a projection procedure is available will be called a *primitive constraint*.
Example. Consider the primitive equation:

\[ x_2 = \sin x_1. \]
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\[ x_2 = \sin x_1 \]
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$x_2 = \sin x_1$
Decomposition

\[ x + \sin(xy) \leq 0, \]
\[ x \in [-1, 1], y \in [-1, 1] \]
Decomposition

\[ x + \sin(xy) \leq 0, \]
\[ x \in [-1, 1], y \in [-1, 1] \]

can be decomposed into

\[
\begin{cases}
  a = xy & x \in [-1, 1] \quad a \in [-\infty, \infty] \\
  b = \sin(a) & y \in [-1, 1] \quad b \in [-\infty, \infty] \\
  c = x + b & c \in [-\infty, 0]
\end{cases}
\]
Forward Backward contractor
For the equation

\[(x_1 + x_2) \cdot x_3 \in [1, 2],\]

we decompose into

\[
\begin{align*}
a &= x_1 + x_2 \\
b &= a \cdot x_3 \\
b &\in [1, 2]
\end{align*}
\]
For the equation

\[(x_1 + x_2) \cdot x_3 \in [1, 2],\]

we have the following contractor:

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>( \mathcal{C} ) (inout([x_1], [x_2], [x_3]))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([a] = [x_1] + [x_2])</td>
<td>(\text{// } a = x_1 + x_2)</td>
</tr>
<tr>
<td>([b] = [a] \cdot [x_3])</td>
<td>(\text{// } b = a \cdot x_3)</td>
</tr>
<tr>
<td>([b] = [b] \cap [1, 2])</td>
<td>(\text{// } b \in [1, 2])</td>
</tr>
<tr>
<td>([x_3] = [x_3] \cap \frac{[b]}{[a]})</td>
<td>(\text{// } x_3 = \frac{b}{a})</td>
</tr>
<tr>
<td>([a] = [a] \cap \frac{[b]}{[x_3]})</td>
<td>(\text{// } a = \frac{b}{x_3})</td>
</tr>
<tr>
<td>([x_1] = [x_1] \cap [a] - [x_2])</td>
<td>(\text{// } x_1 = a - x_2)</td>
</tr>
<tr>
<td>([x_2] = [x_2] \cap [a] - [x_1])</td>
<td>(\text{// } x_2 = a - x_1)</td>
</tr>
</tbody>
</table>
Contractor on images
The robot with coordinates \((x_1, x_2)\) is in the water.
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Solving equations
Consider the system of two equations.

\[ y = x^2 \]
\[ y = \sqrt{x}. \]
We can build two contractors

\[ C_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \text{ associated to } y = x^2 \]

\[ C_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \text{ associated to } y = \sqrt{x} \]
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Another example
Exemple. Consider the system

\[\begin{align*}
y &= 3 \sin(x) \\
y &= x
\end{align*}\]

\(x \in \mathbb{R}, \ y \in \mathbb{R} \).
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Contractor algebra
<table>
<thead>
<tr>
<th>Operation</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intersection</td>
<td>$(\mathcal{C}_1 \cap \mathcal{C}_2)([x]) = \mathcal{C}_1([x]) \cap \mathcal{C}_2([x])$</td>
</tr>
<tr>
<td>Union</td>
<td>$(\mathcal{C}_1 \cup \mathcal{C}_2)([x]) = [\mathcal{C}_1([x]) \cup \mathcal{C}_2([x])]$</td>
</tr>
<tr>
<td>Composition</td>
<td>$(\mathcal{C}_1 \circ \mathcal{C}_2)([x]) = \mathcal{C}_1(\mathcal{C}_2([x]))$</td>
</tr>
<tr>
<td>Repetition</td>
<td>$\mathcal{C}^\infty = \mathcal{C} \circ \mathcal{C} \circ \mathcal{C} \circ \ldots$</td>
</tr>
<tr>
<td>Repeat Intersection</td>
<td>$\mathcal{C}_1 \cap \mathcal{C}_2 = (\mathcal{C}_1 \cap \mathcal{C}_2)^\infty$</td>
</tr>
<tr>
<td>Repeat Union</td>
<td>$\mathcal{C}_1 \cup \mathcal{C}_2 = (\mathcal{C}_1 \cup \mathcal{C}_2)^\infty$</td>
</tr>
</tbody>
</table>

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A link with matrices
\[ \mathcal{L} : \begin{cases} \alpha &= 2a + 3h \\ \gamma &= h - 5a \end{cases} \rightarrow A = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \]

We have a matrix algebra and Matlab.
We have: \( \text{var}(\mathcal{L}) = \{a, h\} \), \( \text{covar}(\mathcal{L}) = \{\alpha, \gamma\} \).
But we cannot write: \( \text{var}(A) = \{a, h\} \), \( \text{covar}(A) = \{\alpha, \gamma\} \).
### Interval analysis

**Contractors**

**Tubes**

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<table>
<thead>
<tr>
<th>constraint</th>
<th>$a \cdot b = z$</th>
</tr>
</thead>
</table>

$\rightarrow$

Diagram of contractors and tubes.
Contractor fusion

\[
\begin{cases}
  a \cdot b = z & \rightarrow \mathcal{C}_1 \\
  b + c = d & \rightarrow \mathcal{C}_2
\end{cases}
\]

Since \( b \) occurs in both constraints, we fuse the two contractors as:

\[
\mathcal{C} = \mathcal{C}_1 \times \mathcal{C}_2 \rangle_{(2,1)} = \mathcal{C}_1 | \mathcal{C}_2 \text{ (for short)}
\]
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SLAM
\[
\begin{align*}
\dot{x} &= f(x, u) \quad \text{(evolution equation)} \\
y &= g(x, u) \quad \text{(observation equation)} \\
z_i &= h(x, u, m_i) \quad \text{(mark equation)}
\end{align*}
\]
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Redermor (GESMA, Brest)

https://youtu.be/X0lqZxb-tFs
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Interval analysis with application to the safe navigation of autonomous vehicles
GPS (Global positioning system), only at the surface.

\begin{align*}
t_0 &= 0000 \text{ s}, \quad & \ell^0 &= (-4.4582279^\circ, 48.2129206^\circ) \pm 2.5m \\
t_f &= 6000 \text{ s}, \quad & \ell^f &= (-4.4546607^\circ, 48.2191297^\circ) \pm 2.5m
\end{align*}
Sonar (KLEIN 5400 side scan sonar).
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Interval analysis with application to the safe navigation of autonomous vehicles.
Screenshot of SonarPro
Mine detection with SonarPro
Loch-Doppler returns the speed robot $v_r$.

\[ v_r \in v_r + 0.004\times[-1,1].v_r + 0.004\times[-1,1] \]
Inertial unit (Octans III).

\[
\begin{pmatrix}
\phi \\
\theta \\
\psi
\end{pmatrix} \in \begin{pmatrix}
\tilde{\phi} \\
\tilde{\theta} \\
\tilde{\psi}
\end{pmatrix} + \begin{pmatrix}
1.75 \times 10^{-4}. [-1, 1] \\
1.75 \times 10^{-4}. [-1, 1] \\
5.27 \times 10^{-3}. [-1, 1]
\end{pmatrix}.
\]
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Six mines have been detected.

<table>
<thead>
<tr>
<th>( i )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tau(i) )</td>
<td>1054</td>
<td>1092</td>
<td>1374</td>
<td>1748</td>
<td>3038</td>
<td>3688</td>
</tr>
<tr>
<td>( \sigma(i) )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>( \tilde{r}(i) )</td>
<td>52.42</td>
<td>12.47</td>
<td>54.40</td>
<td>52.68</td>
<td>27.73</td>
<td>26.98</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>4024</td>
<td>4817</td>
<td>5172</td>
<td>5232</td>
<td>5279</td>
<td>5688</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>37.90</td>
<td>36.71</td>
<td>37.37</td>
<td>31.03</td>
<td>33.51</td>
<td>15.05</td>
</tr>
</tbody>
</table>
Contraint network
\[ t \in \{0, 0.1, 0.2, \ldots, 5999.9\}, \]

\[ i \in \{0, 1, \ldots, 11\}, \]

\[
\begin{pmatrix}
px(t) \\
p_y(t)
\end{pmatrix} = 111120 \begin{pmatrix}
0 & 1 \\
\cos (\ell_y(t) \ast \frac{\pi}{180}) & 0
\end{pmatrix} \begin{pmatrix}
\ell_x(t) - \ell_x^0 \\
\ell_y(t) - \ell_y^0
\end{pmatrix},
\]

\[ \mathbf{p}(t) = (px(t), p_y(t), p_z(t)), \]

\[ R_\psi(t) = \begin{pmatrix}
\cos \psi(t) & -\sin \psi(t) & 0 \\
\sin \psi(t) & \cos \psi(t) & 0 \\
0 & 0 & 1
\end{pmatrix}, \]

\[ R_\theta(t) = \begin{pmatrix}
\cos \theta(t) & 0 & \sin \theta(t) \\
0 & 1 & 0 \\
-\sin \theta(t) & 0 & \cos \theta(t)
\end{pmatrix}, \]
\[
\mathbf{R}_\varphi(t) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \varphi(t) & -\sin \varphi(t) \\
0 & \sin \varphi(t) & \cos \varphi(t)
\end{pmatrix},
\]

\[
\mathbf{R}(t) = \mathbf{R}_\psi(t)\mathbf{R}_\theta(t)\mathbf{R}_\varphi(t),
\]

\[
\mathbf{p}(t) = \mathbf{R}(t)\cdot \mathbf{v}_r(t),
\]

\[
\|\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))\| = r(i),
\]

\[
\mathbf{R}^\top(\tau(i)) (\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))) \in [0] \times [0, \infty]^2,
\]

\[
m_z(\sigma(i)) - p_z(\tau(i)) - a(\tau(i)) \in [-0.5, 0.5]
\]
Interval analysis
Contractors
Tubes

Interval analysis with application to the safe navigation of autonomous vehicles
Interval analysis with application to the safe navigation of autonomous vehicles
Tubes
A trajectory is a function $f : \mathbb{R} \to \mathbb{R}^n$. For instance

$$f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

is a trajectory.
Order relation

\[ f \leq g \iff \forall t, \forall i, f_i(t) \leq g_i(t). \]
We have

\[ h = f \land g \iff \forall t, \forall i, h_i(t) = \min (f_i(t), g_i(t)), \]

\[ h = f \lor g \iff \forall t, \forall i, h_i(t) = \max (f_i(t), g_i(t)). \]
The set of trajectories is a lattice. Interval of trajectories (tubes) can be defined.
Example.

$$[f](t) = \left( \begin{array}{c} \cos t + [0,t^2] \\ \sin t + [-1,1] \end{array} \right)$$

is an interval trajectory (or tube).
Tube arithmetic
If \([x]\) and \([y]\) are two scalar tubes [1], we have

\[
\begin{align*}
[z] &= [x] + [y] \implies [z](t) = [x](t) + [y](t) \\
[z] &= \text{shift}_a([x]) \implies [z](t) = [x](t + a) \\
[z] &= [x] \circ [y] \implies [z](t) = [x]( [y](t)) \\
[z] &= \int [x] \implies [z](t) = \left[ \int_0^t x^- (\tau) \, d\tau, \int_0^t x^+ (\tau) \, d\tau \right]
\end{align*}
\] (sum) (shift) (composition) (integral)
Tube contractor
Tube arithmetic allows us to build contractors.
Consider for instance the differential constraint

\[
\dot{x}(t) = x(t+1) \cdot u(t),
\]
\[
x(t) \in [x](t), \dot{x}(t) \in [\dot{x}](t), u(t) \in [u](t)
\]

We decompose as follows

\[
\begin{cases}
    x(t) &= x(0) + \int_0^t y(\tau) d\tau \\
    y(t) &= a(t) \cdot u(t) \\
    a(t) &= x(t+1)
\end{cases}
\]
Possible contractors are

\[
\begin{align*}
[x](t) &= [x](t) \cap ([x](0) + \int_0^t [y](\tau) d\tau) \\
[y](t) &= [y](t) \cap [a](t) \cdot [u](t) \\
[u](t) &= [u](t) \cap \frac{[y](t)}{[a](t)} \\
[a](t) &= [a](t) \cap \frac{[y](t)}{[u](t)} \\
[a](t) &= [a](t) \cap [x](t + 1) \\
[x](t) &= [x](t) \cap [a](t - 1)
\end{align*}
\]
Example. Consider $x(t) \in [x](t)$ with the constraint
\[ \forall t, \ x(t) = x(t + 1) \]
Contract the tube $[x](t)$.
We first decompose into primitive trajectory constraints

\[ x(t) = a(t+1) \]
\[ x(t) = a(t). \]
Contractors

\[
\begin{align*}
[x](t) & : = [x](t) \cap [a](t + 1) \\
[a](t) & : = [a](t) \cap [x](t - 1) \\
[x](t) & : = [x](t) \cap [a](t) \\
[a](t) & : = [a](t) \cap [x](t)
\end{align*}
\]
Interval analysis
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Space-time estimation
Classical state estimation

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) & t &\in \mathbb{R} \\
0 &= g(x(t), t) & t &\in T \subset \mathbb{R}.
\end{align*}
\]

Space constraint \( g(x(t), t) = 0 \).
Example.

\[
\begin{aligned}
\dot{x}_1 &= x_3 \cos x_4 \\
\dot{x}_2 &= x_3 \cos x_4 \\
\dot{x}_3 &= u_1 \\
\dot{x}_4 &= u_2 \\
(x_1 (5) - 1)^2 + (x_2 (5) - 2)^2 - 4 &= 0 \\
(x_1 (7) - 1)^2 + (x_2 (7) - 2)^2 - 9 &= 0
\end{aligned}
\]
With time-space constraints

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \quad t \in \mathbb{R} \\
0 &= g(x(t), x(t'), t, t') \quad (t, t') \in T \subset \mathbb{R} \times \mathbb{R}.
\end{align*}
\]
**Example.** An ultrasonic underwater robot with state

\[ \mathbf{x} = (x_1, x_2, \ldots) = (x, y, \theta, v, \ldots) \]

At time \( t \) the robot emits an onmidirectional sound. At time \( t' \) it receives it

\[
(x_1(t) - x_1(t'))^2 + (x_2(t) - x_2(t'))^2 - c(t - t')^2 = 0.
\]
Swarm localization
Consider $n$ robots $\mathcal{R}_1, \ldots, \mathcal{R}_n$ described by

$$\dot{x}_i = f(x_i, u_i), u_i \in [u_i].$$
Omnidirectional sounds are emitted and received. 
A ping is a 4-uple \((a, b, i, j)\) where \(a\) is the emission time, \(b\) is the reception time, \(i\) is the emitting robot and \(j\) the receiver.
Interval analysis with application to the safe navigation of autonomous vehicles.
With the time space constraint

\[
\dot{x}_i = f(x_i, u_i), \quad u_i \in [u_i] .
\]

\[
g(x_{i(k)}(a(k)), x_{j(k)}(b(k)), a(k), b(k)) = 0
\]

where

\[
g(x_i, x_j, a, b) = \|x_1 - x_2\| - c(b - a).
\]
Clocks are uncertain. We only have measurements $\tilde{a}(k), \tilde{b}(k)$ of $a(k), b(k)$ thanks to clocks $h_i$. Thus

$$\dot{x}_i = f(x_i, u_i), u_i \in [u_i].$$

$$g \left( x_{i(k)}(a(k)), x_{j(k)}(b(k)), a(k), b(k) \right) = 0$$

$$\tilde{a}(k) = h_{i(k)}(a(k))$$

$$\tilde{b}(k) = h_{j(k)}(b(k))$$
The drift of the clocks is bounded

\[ \dot{x}_i = f(x_i, u_i), u_i \in [u_i]. \]

\[ g(x_{i(k)}(a(k)), x_{j(k)}(b(k)), a(k), b(k)) = 0 \]

\[ \tilde{a}(k) = h_{i(k)}(a(k)) \]

\[ \tilde{b}(k) = h_{j(k)}(b(k)) \]

\[ \dot{h}_i = 1 + n_h, \quad n_h \in [n_h] \]
https://youtu.be/j-ERcoXF1Ks
https://youtu.be/jr8xK1e0Nds
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