Interval analysis with application to the safe navigation of autonomous vehicles

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Interval analysis

Problem. Given $f : \mathbb{R}^n \to \mathbb{R}$ and a box $[\mathbf{x}] \subset \mathbb{R}^n$, prove that

$$\forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic can solve efficiently this problem.

Example. Is the function

$$f(\mathbf{x}) = x_1 x_2 - (x_1 + x_2)\cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

always positive for $x_1, x_2 \in [-1, 1]$?

Interval arithmetic

$$\begin{aligned} [-1,3]+[2,5] &= ?, \\ [-1,3]\cdot[2,5] &= ?, \\ \mathsf{abs}([-7,1]) &= ? \end{aligned}$$

Interval arithmetic

$$
\begin{aligned}
[-1,3]+[2,5] &= [1,8], \\
[-1,3]\cdot[2,5] &= [-5,15], \\
\mathsf{abs}\,([-7,1]) &= [0,7]
\end{aligned}
$$

The interval extension of

$$f(x_1, x_2) = x_1 \cdot x_2 - (x_1 + x_2) \cdot \cos x_2 + \sin x_1 \cdot \sin x_2 + 2$$

is

$$\begin{aligned}[f]([x_1],[x_2]) \quad &= \quad [x_1] \cdot [x_2] - ([x_1] + [x_2]) \cdot \cos[x_2] \\ &\quad + \sin[x_1] \cdot \sin[x_2] + 2.\end{aligned}$$

Theorem (Moore, 1970)

$$[f]([\mathbf{x}]) \subset \mathbb{R}^+ \Rightarrow \forall \mathbf{x} \in [\mathbf{x}], f(\mathbf{x}) \geq 0.$$

Interval arithmetic

If $\diamond \in \{+,-,\cdot,/,\max,\min\}$

$$[x]\diamond[y] \quad = \quad [\ \{x\diamond y \mid x\in[x], y\in[y]\}\ ].$$

where $[\mathbb{A}]$ is the smallest interval which encloses $\mathbb{A}\subset\mathbb{R}$.

Exercise.

$$
\begin{aligned}
[-1,3]+[2,5] &= [?,?] \\
[-1,3]\cdot[2,5] &= [?,?] \\
[-2,6]/[2,5] &= [?,?]
\end{aligned}
$$

Solution.

$$\begin{aligned}
[-1,3]+[2,5] &= [1,8] \\
[-1,3].[2,5] &= [-5,15] \\
[-2,6]/[2,5] &= [-1,3]
\end{aligned}$$

Exercise. Compute

$$[-2,2]/[-1,1] \quad = [?,?]$$

Solution.

$$[-2,2]/[-1,1] \quad = [-\infty, \infty]$$

$$
\begin{aligned}
[x^-,x^+]+[y^-,y^+] = \quad & [x^-+y^-,x^++y^+],\\
[x^-,x^+]\cdot[y^-,y^+] = \quad & [x^-y^-\wedge x^+y^-\wedge x^-y^+\wedge x^+y^+,\\
& x^-y^-\vee x^+y^-\vee x^-y^+\vee x^+y^+],
\end{aligned}
$$

If $f \in \{\cos, \sin, \text{sqr}, \text{sqrt}, \log, \exp, \dots\}$

$$f([x]) = [\{f(x) \mid x \in [x]\}].$$

Exercise.

$$\begin{aligned}
\sin([0,\pi]) &= ? \\
\mathsf{sqr}([-1,3]) &= [-1,3]^2 = ? \\
\mathsf{abs}([-7,1]) &= ? \\
\mathsf{sqrt}([-10,4]) &= \sqrt{[-10,4]} = ? \\
\log([-2,-1]) &= ?.
\end{aligned}$$

Solution.

$$\begin{aligned}
\sin([0,\pi]) &= [0,1] \\
\mathsf{sqr}([-1,3]) &= [-1,3]^2 = [0,9] \\
\mathsf{abs}([-7,1]) &= [0,7] \\
\mathsf{sqrt}([-10,4]) &= \sqrt{[-10,4]} = [0,2] \\
\log([-2,-1]) &= \emptyset.
\end{aligned}$$

Inclusion functions

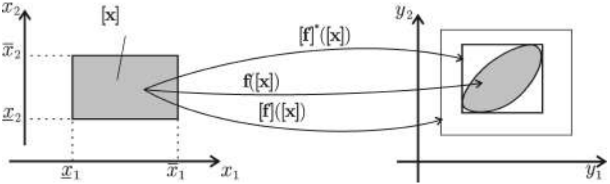
A *box*, or *interval vector* $[\mathbf{x}]$ of $\mathbb{R}^n$ is

$$[\mathbf{x}] = [x_1^-, x_1^+] \times \cdots \times [x_n^-, x_n^+] = [x_1] \times \cdots \times [x_n].$$

The set of all boxes of $\mathbb{R}^n$ will be denoted by $\mathbb{IR}^n$.

$[\mathbf{f}] : \mathbb{IR}^n \to \mathbb{IR}^m$ is an *inclusion function* for $\mathbf{f}$ if

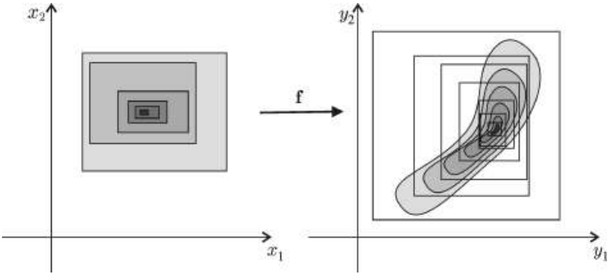
$$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \ \mathbf{f}([\mathbf{x}]) \subset [\mathbf{f}]\,([\mathbf{x}]).$$



Inclusion functions $[\mathbf{f}]$ and $[\mathbf{f}]^*$; here, $[\mathbf{f}]^*$ is minimal.

The inclusion function $[\mathbf{f}]$ is

monotonic	if	$([\mathbf{x}] \subset [\mathbf{y}]) \Rightarrow ([\mathbf{f}]([\mathbf{x}]) \subset [\mathbf{f}]([\mathbf{y}]))$
minimal	if	$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ [\mathbf{f}]([\mathbf{x}]) = [\mathbf{f}([\mathbf{x}])]$
thin	if	$w([\mathbf{x}]) = 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) = 0$
convergent	if	$w([\mathbf{x}]) \to 0 \Rightarrow w([\mathbf{f}]([\mathbf{x}]) \to 0.$


x_2
x_1
f
y_2
y_1

Exercise. The natural inclusion function for $f(x) = x^2 + 2x + 4$ is

$$[f]([x]) = [x]^2 + 2[x] + 4.$$

For $[x] = [-3, 4]$, compute $[f]([x])$ and $f([x])$.

Solution. If $[x] = [-3,4]$, we have

$$\begin{aligned} [f]([-3,4]) &= [-3,4]^2 + 2[-3,4] + 4 \\ &= [0,16] + [-6,8] + 4 \\ &= [-2,28]. \end{aligned}$$

Note that $f([-3,4]) = [3,28] \subset [f]([-3,4]) = [-2,28]$.

A minimal inclusion function for

$$\mathbf{f}: \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^3 \\ (x_1, x_2) & \mapsto & \left(x_1 x_2, x_1^2, x_1 - x_2\right). \end{array}$$

is

$$[\mathbf{f}]: \begin{array}{ccc} \mathbb{IR}^2 & \to & \mathbb{IR}^3 \\ ([x_1],[x_2]) & \to & \left([x_1]\cdot[x_2],[x_1]^2,[x_1]-[x_2]\right). \end{array}$$

If $\mathbf{f}$ is given by

Algorithm $\mathbf{f}(\text{in} : \mathbf{x} = (x_1, x_2, x_3),\ \text{out} : \mathbf{y} = (y_1, y_2))$
$z := x_1$
for $k := 0$ to 100
$\quad\quad z := x_2(z + k \cdot x_3)$
next
$y_1 := z$
$y_2 := \sin(zx_1)$

Its natural inclusion function is

Algorithm $[\mathbf{f}]$(in : $[\mathbf{x}] = ([x_1],[x_2],[x_3])$, out : $[\mathbf{y}] = ([y_1],[y_2])$)
$[z] := [x_1]$ for$k := 0$ to 100 $[z] := [x_2] \cdot ([z] + k \cdot [x_3])$ next $[y_1] := [z]$ $[y_2] := \sin([z] \cdot [x_1])$

Is $[\mathbf{f}]$ convergent? thin? monotonic?

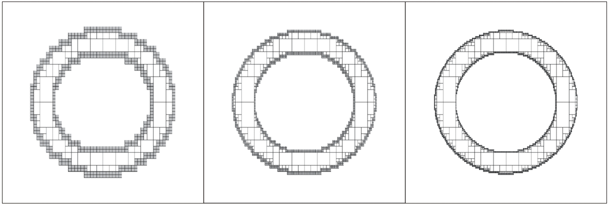
Set inversion

A subpaving of $\mathbb{R}^n$ is a set of non-overlapping boxes of $\mathbb{R}^n$. Compact sets $\mathbb{X}$ can be bracketed between inner and outer subpavings:

$$\mathbb{X}^- \subset \mathbb{X} \subset \mathbb{X}^+.$$

Example.

$$\mathbb{X} = \{(x_1, x_2) \mid x_1^2 + x_2^2 \in [1,2]\}.$$



Let $\mathbf{f}:\mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbb{Y}$ be a subset of $\mathbb{R}^m$. Set inversion is the characterization of

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{f}(\mathbf{x}) \in \mathbb{Y}\} = \mathbf{f}^{-1}(\mathbb{Y}).$$

We shall use the following tests.

$$\begin{array}{lllll}\text{(i)} & [\mathbf{f}]([\mathbf{x}]) \subset \mathbb{Y} & \Rightarrow & [\mathbf{x}] \subset \mathbb{X} \\ \text{(ii)} & [\mathbf{f}]([\mathbf{x}]) \cap \mathbb{Y} = \emptyset & \Rightarrow & [\mathbf{x}] \cap \mathbb{X} = \emptyset. \end{array}$$

Boxes for which these tests failed, will be bisected, except if they are too small.

Localization

A robot measures distances to three beacons.

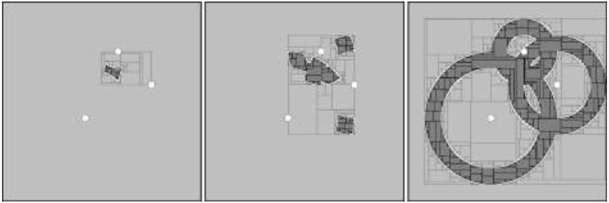
i	x_i	y_i	$[d_i]$
1	1	3	$[1,2]$
2	3	1	$[2,3]$
3	-1	-1	$[3,4]$

The intervals $[d_i]$ contain the true distance with a probability of $\pi = 0.9$.

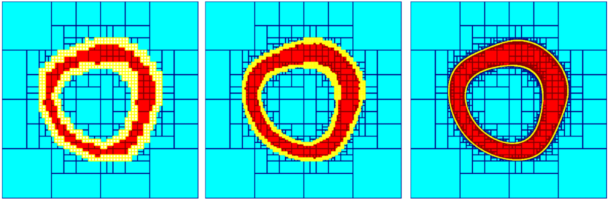
Define

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \mid \sqrt{(p_1 - x_i)^2 + (p_2 - y_i)^2} \in [d_i] \right\}.$$

$$\begin{aligned}
\text{prob}\left(\mathbf{p}\in\mathbb{P}^{\{0\}}\right) &= 0.729\\
\text{prob}\left(\mathbf{p}\in\mathbb{P}^{\{1\}}\right) &= 0.972\\
\text{prob}\left(\mathbf{p}\in\mathbb{P}^{\{2\}}\right) &= 0.999
\end{aligned}$$



Contractors

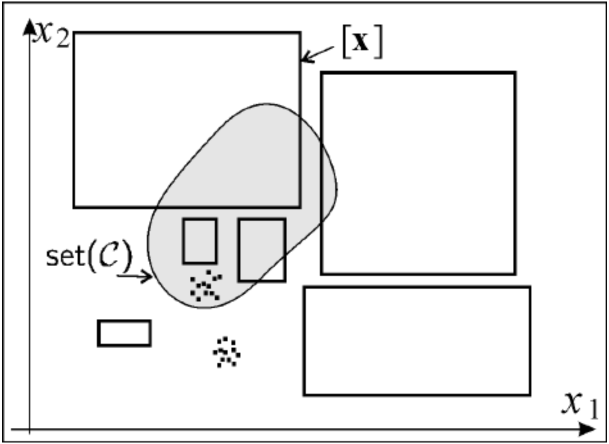


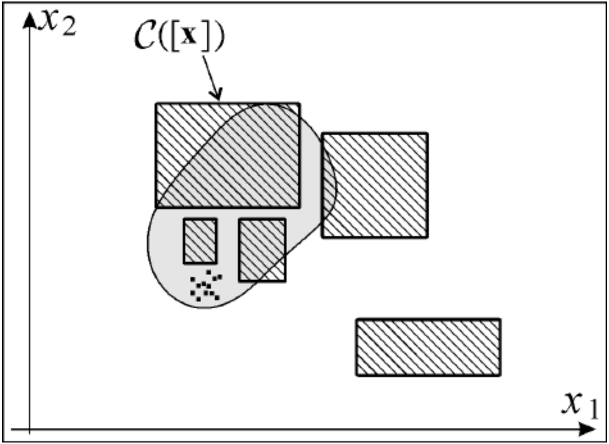
To characterize $\mathbb{X} \subset \mathbb{R}^n$, bisection algorithms bisect all boxes in all directions and become inefficient. Interval methods can still be useful if

- the solution set $\mathbb{X}$ is small (optimization problem, solving equations),
- contraction procedures are used as much as possible,
- bisections are used only as a last resort.

The operator $\mathscr{C}:\mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for $\mathbb{X} \subset \mathbb{R}^n$ if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \begin{cases} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] & \text{(contractance),} \\ \mathscr{C}([\mathbf{x}]) \cap \mathbb{X} = [\mathbf{x}] \cap \mathbb{X} & \text{(completeness).} \end{cases}$$


x_2
$[\mathbf{x}]$
set($\mathcal{C}$)
x_1


x_2
$\mathcal{C}([\mathbf{x}])$
x_1

The operator $\mathscr{C} : \mathbb{IR}^n \to \mathbb{IR}^n$ is a *contractor* for the equation $f(\mathbf{x}) = 0$, if

$$\forall [\mathbf{x}] \in \mathbb{IR}^n, \left\{ \begin{array}{l} \mathscr{C}([\mathbf{x}]) \subset [\mathbf{x}] \\ \mathbf{x} \in [\mathbf{x}] \text{ et } f(\mathbf{x}) = 0 \Rightarrow \mathbf{x} \in \mathscr{C}([\mathbf{x}]) \end{array} \right.$$

Exercice. Let x, y, z be 3 variables such that

$$\begin{array}{rcl} x & \in & [-\infty, 5], \\ y & \in & [-\infty, 4], \\ z & \in & [6, \infty], \\ z & = & x + y. \end{array}$$

Contract the intervals for x, y, z.

Solution.

$$\begin{aligned}[x] &= [2,5]\\ [y] &= [1,4]\\ [z] &= [6,9]\end{aligned}$$

Since $x \in [-\infty, 5], y \in [-\infty, 4], z \in [6, \infty]$ and $z = x + y$, we have

$$\begin{array}{rll} z = x + y \Rightarrow & z \in & [6,\infty] \cap ([-\infty,5] + [-\infty,4]) \\ & & = [6,\infty] \cap [-\infty,9] = [6,9]. \\ x = z - y \Rightarrow & x \in & [-\infty,5] \cap ([6,\infty] - [-\infty,4]) \\ & & = [-\infty,5] \cap [2,\infty] = [2,5]. \\ y = z - x \Rightarrow & y \in & [-\infty,4] \cap ([6,\infty] - [-\infty,5]) \\ & & = [-\infty,4] \cap [1,\infty] = [1,4]. \end{array}$$

The contractor associated with $z = x + y$ is:

Algorithm pplus(inout: $[z], [x], [y]$)

$[z] := [z] \cap ([x] + [y])$	// $z = x + y$
$[x] := [x] \cap ([z] - [y])$	// $x = z - y$
$[y] := [y] \cap ([z] - [x])$	// $y = z - x$

The contractor associated with $z = x \cdot y$ is:

Algorithm pmult (inout: $[z], [x], [y]$)

$[z] := [z] \cap ([x] \cdot [y])$ // $z = x \cdot y$
$[x] := [x] \cap ([z] \cdot 1/[y])$ // $x = z/y$
$[y] := [y] \cap ([z] \cdot 1/[x])$ // $y = z/x$

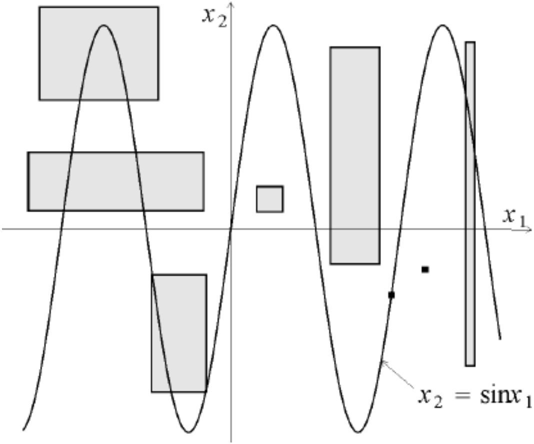
The contractor associated with $y = \exp x$ is:

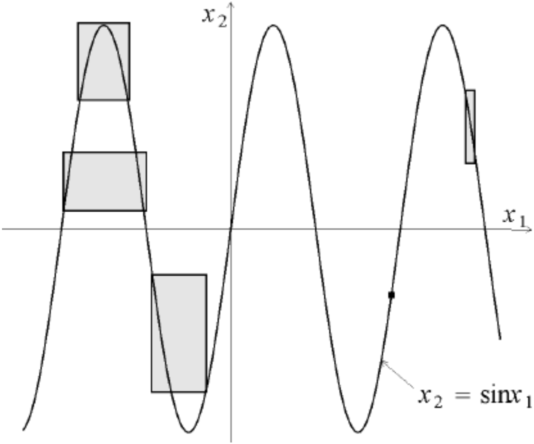
	Algorithm pexp (inout: $[y], [x]$)
1	$[y] := [y] \cap \exp([x])$
2	$[x] := [x] \cap \log([y])$

Any constraint for which such a projection procedure is available will be called a *primitive constraint*.

Example. Consider the primitive equation:

$$x_2 = \sin x_1.$$


x_2
x_1
$x_2 = \sin x_1$


x_2
x_1
$x_2 = \sin x_1$

Decomposition

$$x + \sin(xy) \leq 0,$$
$$x \in [-1,1], y \in [-1,1]$$

Decomposition

$$x + \sin(xy) \leq 0,$$
$$x \in [-1,1], y \in [-1,1]$$

can be decomposed into

$$\left\{ \begin{array}{l} a = xy \\ b = \sin(a) \\ c = x + b \end{array} \right. , \qquad \begin{array}{ll} x \in [-1,1] & a \in [-\infty,\infty] \\ y \in [-1,1] & b \in [-\infty,\infty] \\ & c \in [-\infty,0] \end{array}$$

Forward Backward contractor

For the equation

$$(x_1 + x_2) \cdot x_3 \in [1, 2],$$

we decompose into

$$\begin{array}{rcl} a & = & x_1 + x_2 \\ b & = & a \cdot x_3 \\ b & \in & [1, 2] \end{array}$$

For the equation

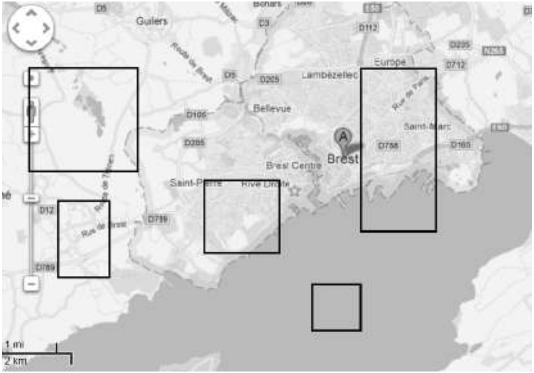
$$(x_1 + x_2) \cdot x_3 \in [1, 2],$$

we have the following contractor:

Algorithm $\mathscr{C}$ (inout$[x_1], [x_2], [x_3]$)	
$[a] = [x_1] + [x_2]$	// $a = x_1 + x_2$
$[b] = [a] \cdot [x_3]$	// $b = a \cdot x_3$
$[b] = [b] \cap [1, 2]$	// $b \in [1, 2]$
$[x_3] = [x_3] \cap \frac{[b]}{[a]}$	// $x_3 = \frac{b}{a}$
$[a] = [a] \cap \frac{[b]}{[x_3]}$	// $a = \frac{b}{x_3}$
$[x_1] = [x_1] \cap [a] - [x_2]$	// $x_1 = a - x_2$
$[x_2] = [x_2] \cap [a] - [x_1]$	// $x_2 = a - x_1$

Contractor on images

The robot with coordinates (x_1, x_2) is in the water.


Guilers
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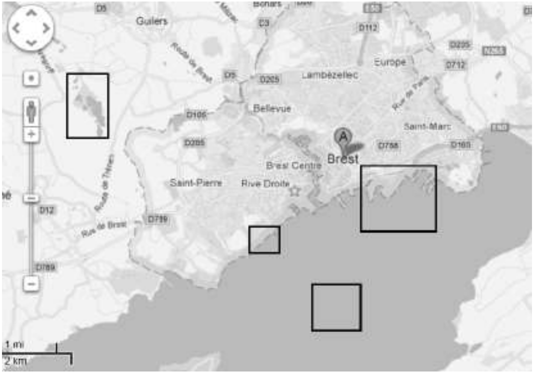
Europe
Bellevue
Saint-Marc
Brest
Brest Centre
Saint-Pierre
Rive Droite
1 mi
2 km


Guilers
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Saint-Pierre
Rive Droite
Route de Brest
Rue de Brest
Rue de Paris
D12
D789
D205
D112
D712
D788
1 mi
2 km

Solving equations

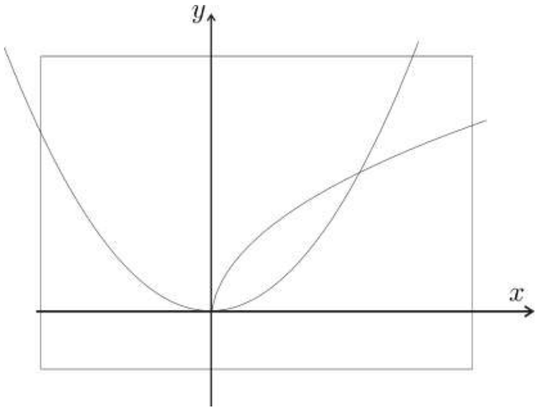
Consider the system of two equations.

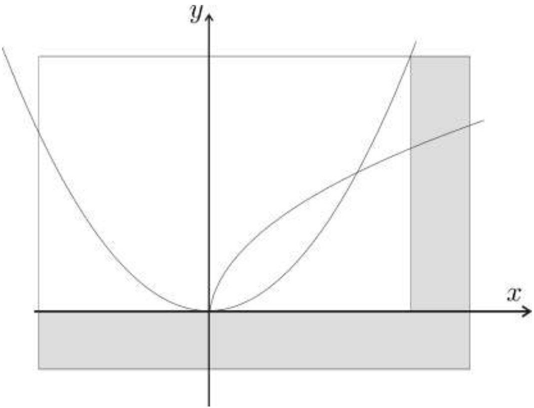
$$\begin{aligned} y &= x^2 \\ y &= \sqrt{x}. \end{aligned}$$

We can build two contractors

$$\mathscr{C}_1 : \begin{cases} [y] = [y] \cap [x]^2 \\ [x] = [x] \cap \sqrt{[y]} \end{cases} \quad \text{associated to } y = x^2$$

$$\mathscr{C}_2 : \begin{cases} [y] = [y] \cap \sqrt{[x]} \\ [x] = [x] \cap [y]^2 \end{cases} \quad \text{associated to } y = \sqrt{x}$$


y
x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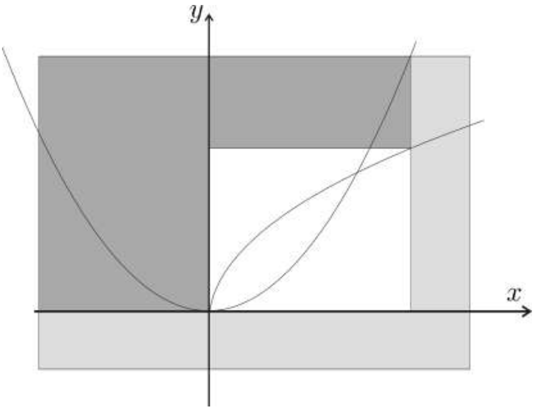
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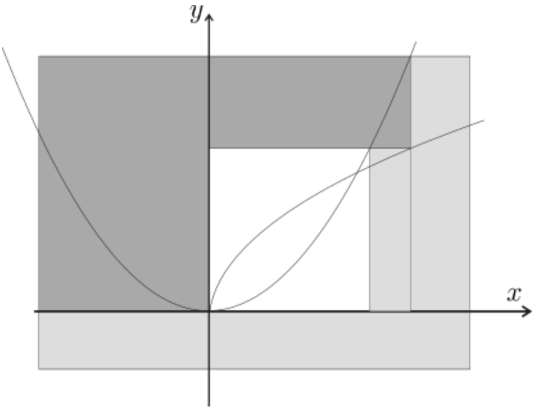
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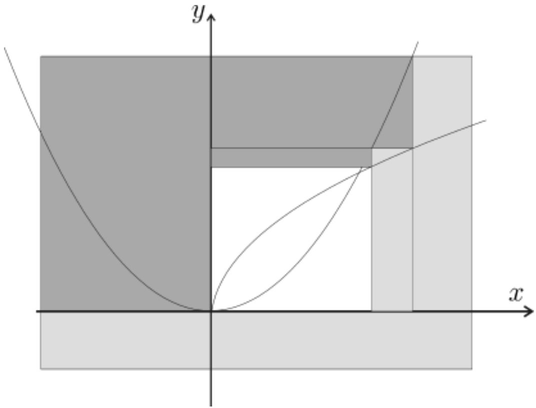
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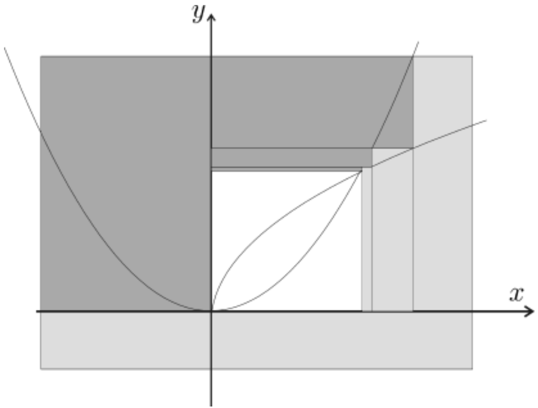
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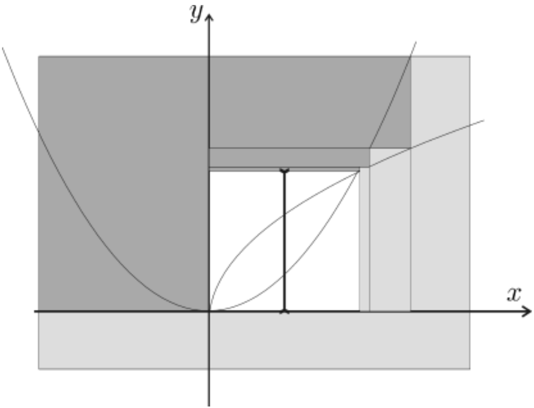
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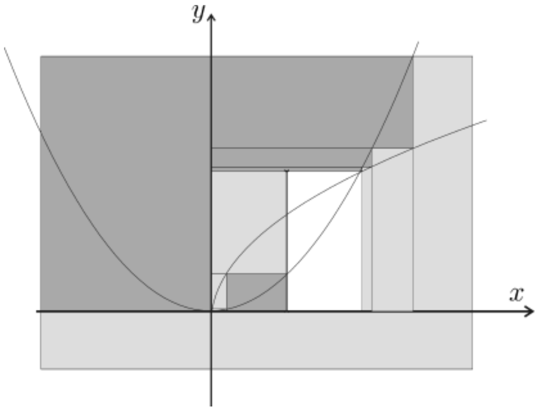
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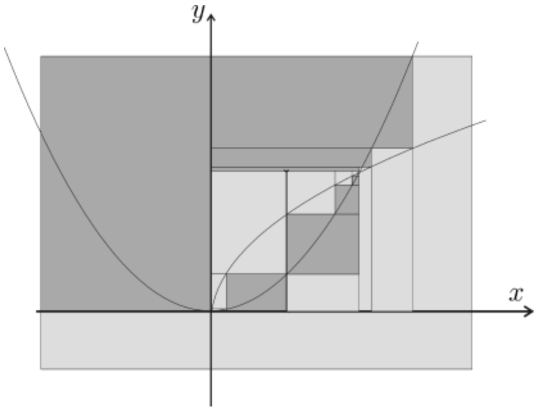
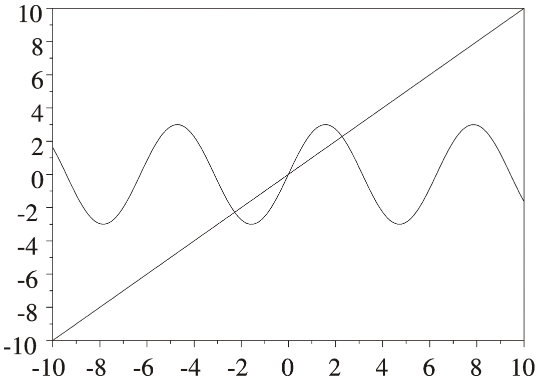

y
x

Another example

Exemple. Consider the system

$$\left\{ \begin{array}{rcl} y & = & 3\sin(x) \\ y & = & x \end{array} \right. \qquad x \in \mathbb{R},\ y \in \mathbb{R}.$$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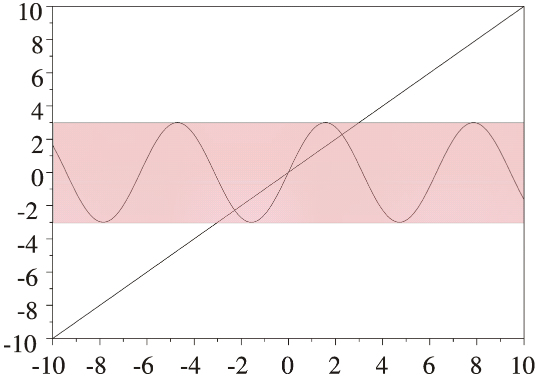
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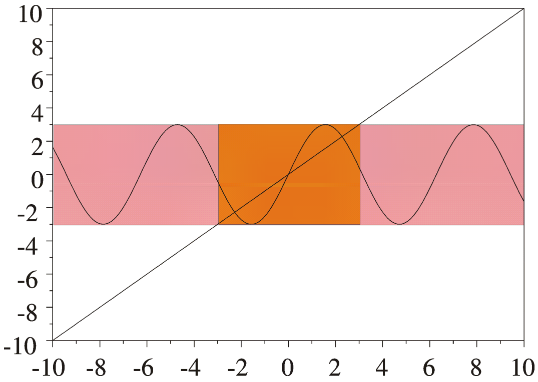
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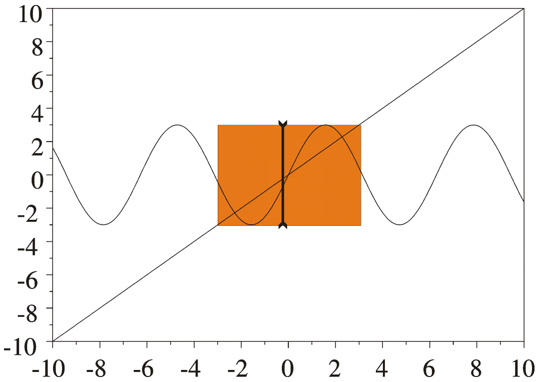
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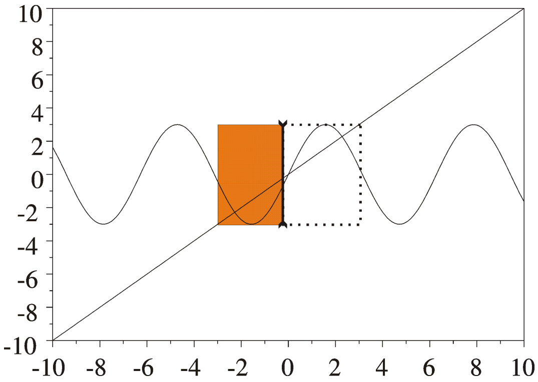
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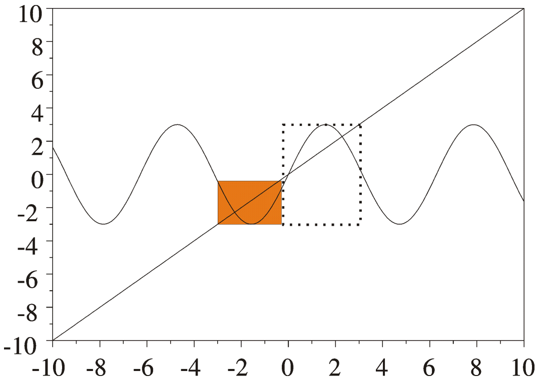
10
8
6
4
2
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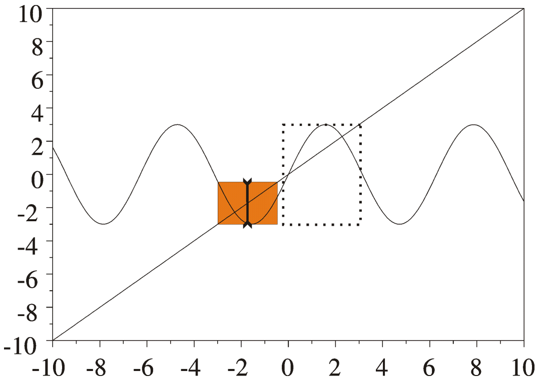
-2
-4
-6
-8
-10
-10
-8
-6
-4
-2
0
2
4
6
8
10


10
8
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-2
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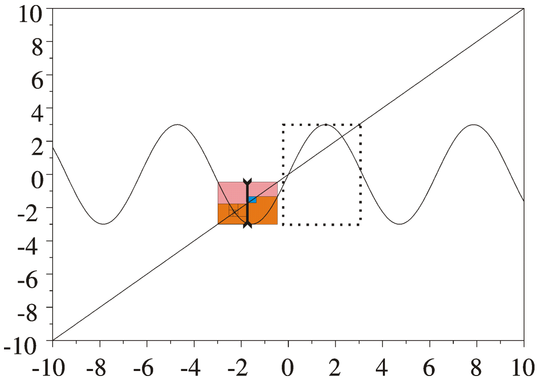
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10


10
8
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-6
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-8
-6
-4
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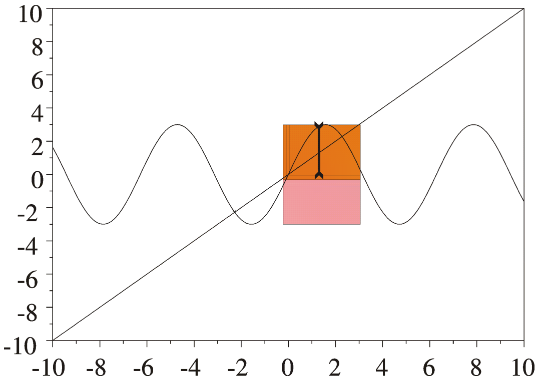
0
2
4
6
8
10


10
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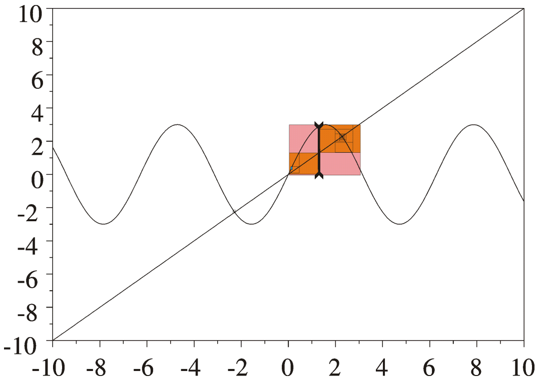
-10 -8 -6 -4 -2 0 2 4 6 8 10


10
8
6
4
2
0
-2
-4
-6
-8
-10
-10 -8 -6 -4 -2 0 2 4 6 8 10

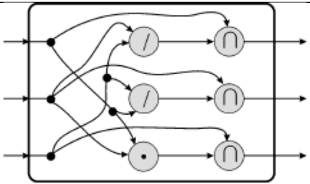
Contractor algebra

intersection	$(\mathscr{C}_1 \cap \mathscr{C}_2)([\mathbf{x}]) = \mathscr{C}_1([\mathbf{x}]) \cap \mathscr{C}_2([\mathbf{x}])$
union	$(\mathscr{C}_1 \cup \mathscr{C}_2)([\mathbf{x}]) = [\mathscr{C}_1([\mathbf{x}]) \cup \mathscr{C}_2([\mathbf{x}])]$
composition	$(\mathscr{C}_1 \circ \mathscr{C}_2)([\mathbf{x}]) = \mathscr{C}_1(\mathscr{C}_2([\mathbf{x}]))$
repetition	$\mathscr{C}^\infty = \mathscr{C} \circ \mathscr{C} \circ \mathscr{C} \circ \dots$
repeat intersection	$\mathscr{C}_1 \sqcap \mathscr{C}_2 = (\mathscr{C}_1 \cap \mathscr{C}_2)^\infty$
repeat union	$\mathscr{C}_1 \sqcup \mathscr{C}_2 = (\mathscr{C}_1 \cup \mathscr{C}_2)^\infty$

A link with matrices

$$\begin{array}{ccc} \text{linear application} & \rightarrow & \text{matrices} \\ \mathscr{L}:\left\{\begin{array}{ccc} \alpha & = & 2a+3h \\ \gamma & = & h-5a \end{array}\right. & \rightarrow & \mathbf{A}=\left(\begin{array}{cc} 2 & 3 \\ 1 & -5 \end{array}\right) \end{array}$$

We have a matrix algebra and Matlab.
We have: $\text{var}(\mathscr{L})=\{a,h\}$, $\text{covar}(\mathscr{L})=\{\alpha,\gamma\}$.
But we cannot write: $\text{var}(\mathbf{A})=\{a,h\}$, $\text{covar}(\mathbf{A})=\{\alpha,\gamma\}$.

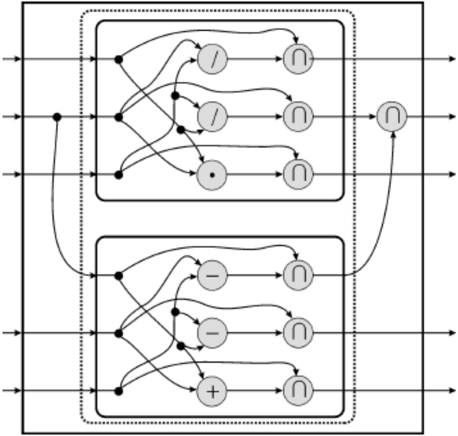
constraint	$\rightarrow$	contractor
$a \cdot b = z$	$\rightarrow$	

Contractor fusion

$$\left\{ \begin{array}{lcl} a \cdot b = z & \rightarrow & \mathscr{C}_1 \\ b + c = d & \rightarrow & \mathscr{C}_2 \end{array} \right.$$

Since b occurs in both constraints, we fuse the two contractors as:

$$\begin{array}{rcl} \mathscr{C} & = & \mathscr{C}_1 \times \mathscr{C}_2 \rfloor_{(2,1)} \\ & = & \mathscr{C}_1 | \mathscr{C}_2 \text{ (for short)} \end{array}$$



SLAM

$$\begin{cases} \dot{\mathbf{x}} & = & \mathbf{f}(\mathbf{x},\mathbf{u}) & \text{(evolution equation)} \\ \mathbf{y} & = & \mathbf{g}(\mathbf{x},\mathbf{u}) & \text{(observation equation)} \\ \mathbf{z}_i & = & \mathbf{h}(\mathbf{x},\mathbf{u},\mathbf{m}_i) & \text{(mark equation)} \end{cases}$$

Redermor (GESMA, Brest)

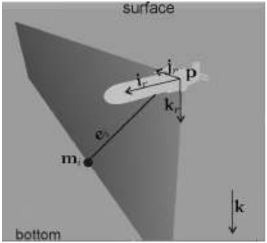
https://youtu.be/X0IqZxb-tFs

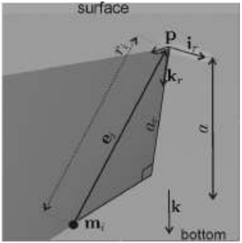

A785

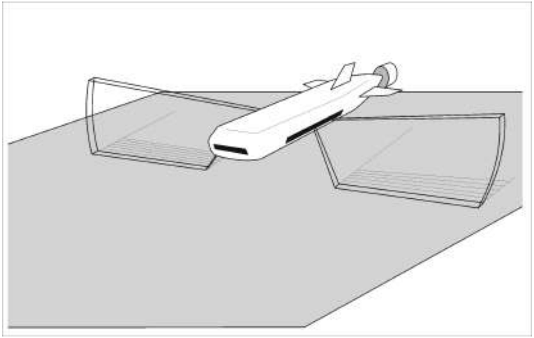
GPS (Global positioning system), only at the surface.

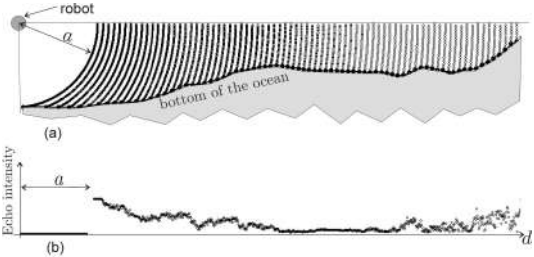
$$t_0 = 0000 \text{ s}, \quad \ell^0 = (-4.4582279°, 48.2129206°) \pm 2.5m$$
$$t_f = 6000 \text{ s}, \quad \ell^f = (-4.4546607°, 48.2191297°) \pm 2.5m$$

Sonar (KLEIN 5400 side scan sonar).

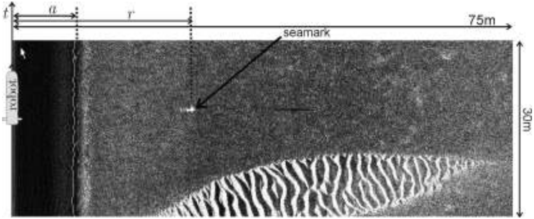

surface
$\mathbf{j}_r$
$\mathbf{p}$
$\mathbf{i}_r$
$\mathbf{k}_r$
$\mathbf{e}_i$
$\mathbf{m}_i$
$\mathbf{k}$
bottom


surface
$\mathbf{p}$
$\mathbf{i}_r$
$\mathbf{k}_r$
$\mathbf{e}_i$
$\mathbf{k}$
$\mathbf{m}_i$
bottom




robot
a
bottom of the ocean
(a)
Echo intensity
a
d
(b)

Screenshot of SonarPro



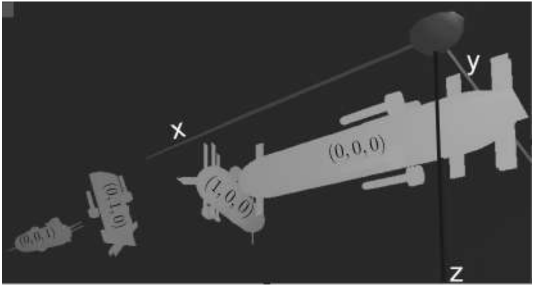
Mine detection with SonarPro

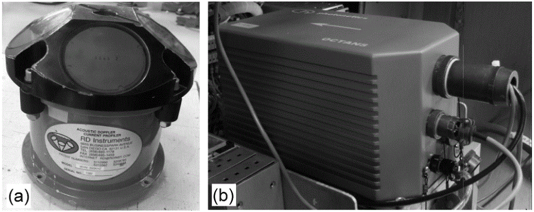
Loch-Doppler returns the speed robot $\mathbf{v}_r$.

$$\mathbf{v}_r \in \underset{\sim}{\mathbf{v}}_r + 0.004 * [-1,1] . \underset{\sim}{\mathbf{v}}_r + 0.004 * [-1,1]$$

Inertial unit (Octans III).

$$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} \in \begin{pmatrix} \tilde{\phi} \\ \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} + \begin{pmatrix} 1.75 \times 10^{-4}.[-1,1] \\ 1.75 \times 10^{-4}.[-1,1] \\ 5.27 \times 10^{-3}.[-1,1] \end{pmatrix}.$$


x
y
z
(0,0,0)
(1,0,0)
(0,1,0)


(a)
ACOUSTIC DOPPLER
CURRENT PROFILER
RD Instruments
(b)
OCTANS

Six mines have been detected.

i	0	1	2	3	4	5	6	7	8	9	10	11
$\tau(i)$	1054	1092	1374	1748	3038	3688	4024	4817	5172	5232	5279	5688
$\sigma(i)$	1	2	1	0	1	5	4	3	3	4	5	1
$\tilde{r}(i)$	52.42	12.47	54.40	52.68	27.73	26.98	37.90	36.71	37.37	31.03	33.51	15.05

Contraint network

$$t \in \{.0, 0.1, 0.2, \ldots, 5999.9\},$$

$$i \in \{0, 1, \ldots, 11\},$$

$$\begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} = 111120 \begin{pmatrix} 0 & 1 \\ \cos\left(\ell_y(t) * \frac{\pi}{180}\right) & 0 \end{pmatrix} \begin{pmatrix} \ell_x(t) - \ell_x^0 \\ \ell_y(t) - \ell_y^0 \end{pmatrix},$$

$$\mathbf{p}(t) = (p_x(t), p_y(t), p_z(t)),$$

$$\mathbf{R}_\psi(t) = \begin{pmatrix} \cos\psi(t) & -\sin\psi(t) & 0 \\ \sin\psi(t) & \cos\psi(t) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{R}_\theta(t) = \begin{pmatrix} \cos\theta(t) & 0 & \sin\theta(t) \\ 0 & 1 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) \end{pmatrix},$$

$$\mathbf{R}_{\varphi}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi(t) & -\sin\varphi(t) \\ 0 & \sin\varphi(t) & \cos\varphi(t) \end{pmatrix},$$

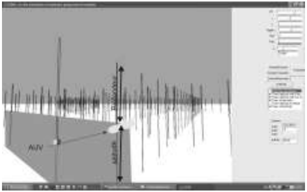
$$\mathbf{R}(t) = \mathbf{R}_{\psi}(t)\mathbf{R}_{\theta}(t)\mathbf{R}_{\varphi}(t),$$

$$\dot{\mathbf{p}}(t) = \mathbf{R}(t).\mathbf{v}_r(t),$$

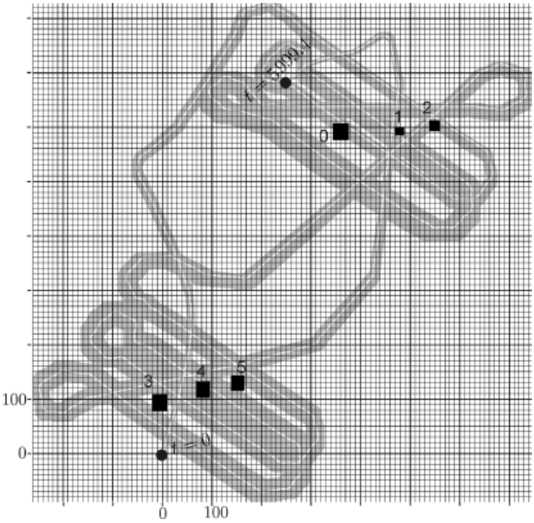
$$||\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))|| \ = r(i),$$

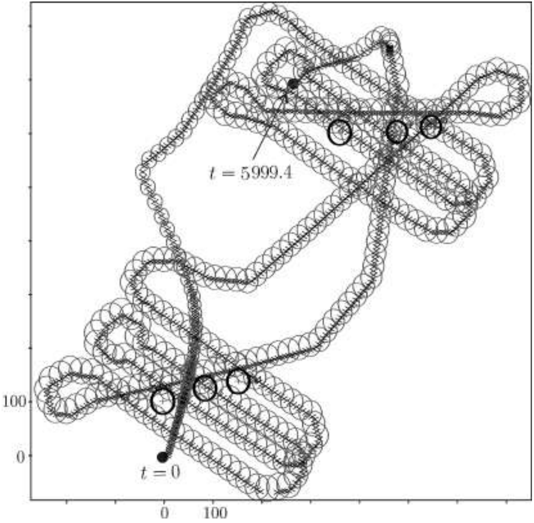
$$\mathbf{R}^{\mathsf{T}}(\tau(i))\,(\mathbf{m}(\sigma(i)) - \mathbf{p}(\tau(i))) \in [0] \times [0,\infty]^{\times 2},$$

$$m_z(\sigma(i)) - p_z(\tau(i)) - a(\tau(i)) \in [-0.5, 0.5]$$



youtu.be/lzJtAfAT7h4


t = 5999.4
0
1
2
3
4
5
100
0
t = 0
0
100


t = 5999.4
100
0
t = 0
0
100

Tubes

A trajectory is a function $\mathbf{f}:\mathbb{R}\to\mathbb{R}^n$. For instance

$$\mathbf{f}(t)=\left(\begin{array}{c}\cos t\\ \sin t\end{array}\right)$$

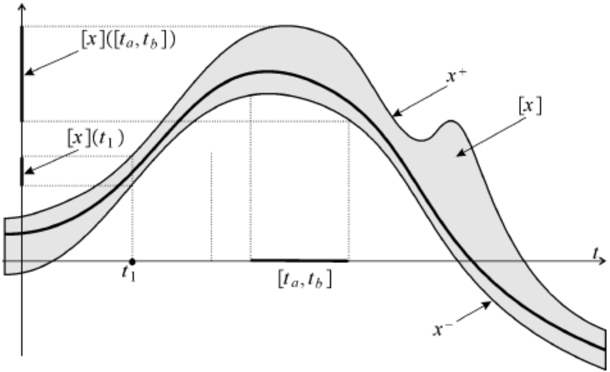
is a trajectory.

Order relation

$$\mathbf{f} \leq \mathbf{g} \Leftrightarrow \forall t, \forall i, f_i(t) \leq g_i(t).$$

We have

$$\begin{aligned} \mathbf{h} &= \mathbf{f} \wedge \mathbf{g} \Leftrightarrow \forall t, \forall i, h_i(t) = \min\left(f_i(t), g_i(t)\right), \\ \mathbf{h} &= \mathbf{f} \vee \mathbf{g} \Leftrightarrow \forall t, \forall i, h_i(t) = \max\left(f_i(t), g_i(t)\right). \end{aligned}$$



The set of trajectories is a lattice. Interval of trajectories (tubes) can be defined.

Example.

$$[\mathbf{f}]\,(t) = \begin{pmatrix} \cos t + [0, t^2] \\ \sin t + [-1, 1] \end{pmatrix}$$

is an interval trajectory (or tube).

Tube arithmetic

If $[x]$ and $[y]$ are two scalar tubes [1], we have

$$
\begin{array}{ll}
[z]=[x]+[y]\Rightarrow[z](t)=[x](t)+[y](t) & \text{(sum)}\\
[z]=\mathrm{shift}_a([x])\Rightarrow[z](t)=[x](t+a) & \text{(shift)}\\
[z]=[x]\circ[y]\Rightarrow[z](t)=[x]([y](t)) & \text{(composition)}\\
[z]=\int[x]\Rightarrow[z](t)=\left[\int_0^t x^-(\tau)\,d\tau,\int_0^t x^+(\tau)\,d\tau\right] & \text{(integral)}
\end{array}
$$

Tube contractor

Tube arithmetic allows us to build contractors.

Consider for instance the differential constraint

$$\begin{array}{rcl} \dot{x}(t) & = & x(t+1)\cdot u(t), \\ x(t) & \in & [x](t), \dot{x}(t) \in [\dot{x}](t), u(t) \in [u](t) \end{array}$$

We decompose as follows

$$\left\{ \begin{array}{rcl} x(t) & = & x(0) + \int_0^t y(\tau)\, d\tau \\ y(t) & = & a(t)\cdot u(t). \\ a(t) & = & x(t+1) \end{array} \right.$$

Possible contractors are

$$\left\{ \begin{array}{lcl} [x](t) & = & [x](t) \cap \left([x](0) + \int_0^t [y](\tau)\, d\tau\right) \\ [y](t) & = & [y](t) \cap [a](t) \cdot [u](t) \\ [u](t) & = & [u](t) \cap \frac{[y](t)}{[a](t)} \\ [a](t) & = & [a](t) \cap \frac{[y](t)}{[u](t)} \\ [a](t) & = & [a](t) \cap [x](t+1) \\ [x](t) & = & [x](t) \cap [a](t-1) \end{array} \right.$$

Example. Consider $x(t) \in [x](t)$ with the constraint

$$\forall t,\ x(t) = x(t+1)$$

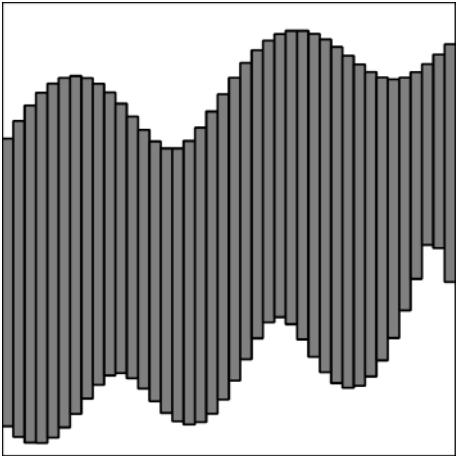
Contract the tube $[x](t)$.

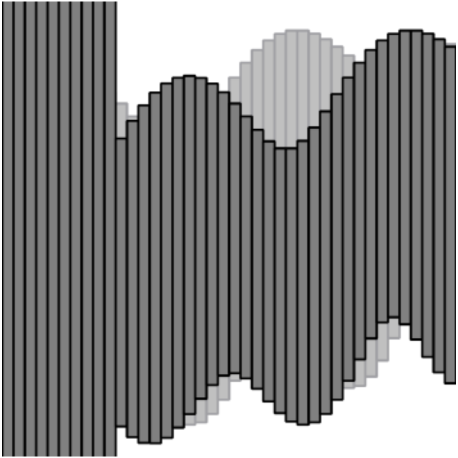
We first decompose into primitive trajectory constraints

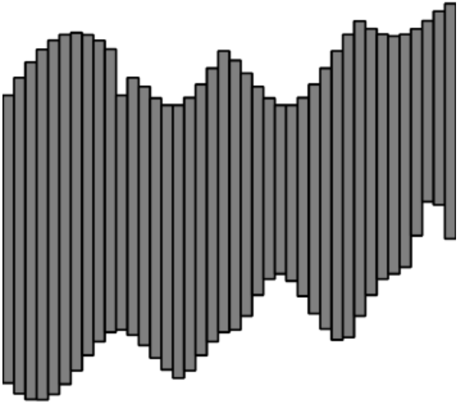
$$\begin{array}{rcl} x(t) & = & a(t+1) \\ x(t) & = & a(t). \end{array}$$

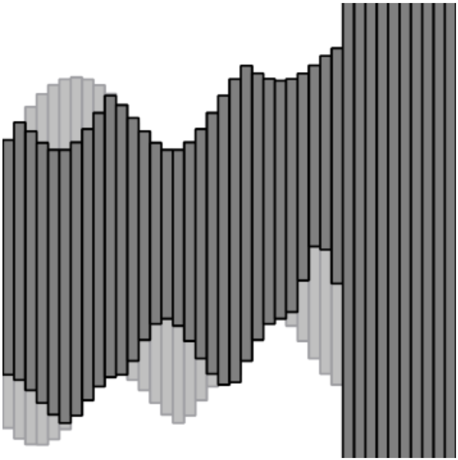
Contractors

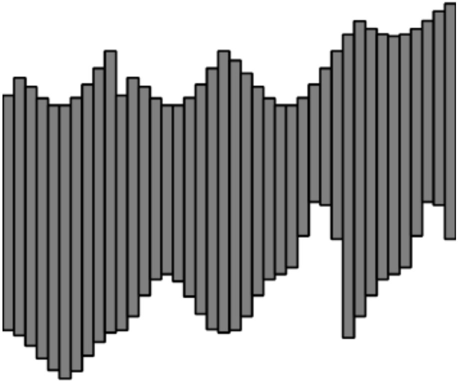
$$
\begin{aligned}
[x](t) &:= [x](t)\cap[a](t+1)\\
[a](t) &:= [a](t)\cap[x](t-1)\\
[x](t) &:= [x](t)\cap[a](t)\\
[a](t) &:= [a](t)\cap[x](t)
\end{aligned}
$$

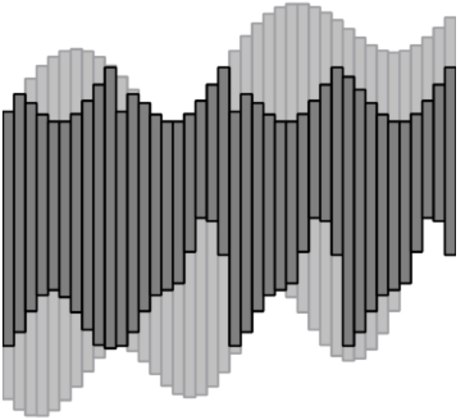


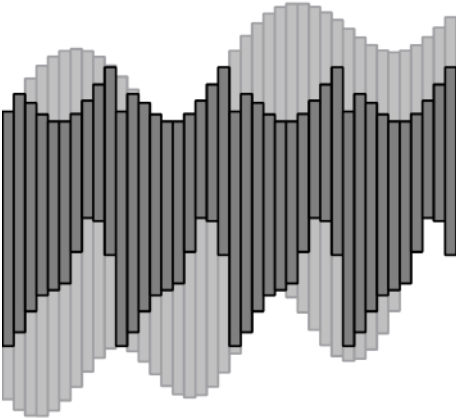


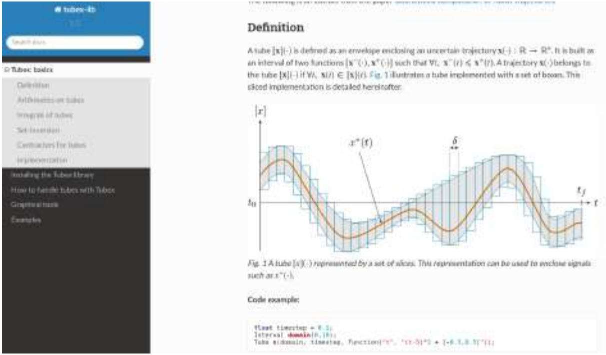












http://www.codac.io/

Space-time estimation

Classical state estimation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & t \in \mathbb{R} \\ \mathbf{0} = \mathbf{g}(\mathbf{x}(t), t) & t \in \mathbb{T} \subset \mathbb{R}. \end{cases}$$

Space constraint $\mathbf{g}(\mathbf{x}(t), t) = 0$.

Example.

$$\begin{cases} \dot{x}_1 = x_3 \cos x_4 \\ \dot{x}_2 = x_3 \cos x_4 \\ \dot{x}_3 = u_1 \\ \dot{x}_4 = u_2 \\ (x_1(5) - 1)^2 + (x_2(5) - 2)^2 - 4 = 0 \\ (x_1(7) - 1)^2 + (x_2(7) - 2)^2 - 9 = 0 \end{cases}$$

With time-space constraints

$$\left\{\begin{array}{lllll}\dot{\mathbf{x}}(t) & = & \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) & & t\in\mathbb{R}\\ \mathbf{0} & = & \mathbf{g}(\mathbf{x}(t),\mathbf{x}(t'),t,t') & & (t,t')\in\mathbb{T}\subset\mathbb{R}\times\mathbb{R}.\end{array}\right.$$

Example. An ultrasonic underwater robot with state

$$\mathbf{x} = (x_1, x_2, \dots) = (x, y, \theta, v, \dots)$$

At time t the robot emits an onmidirectional sound. At time t' it receives it

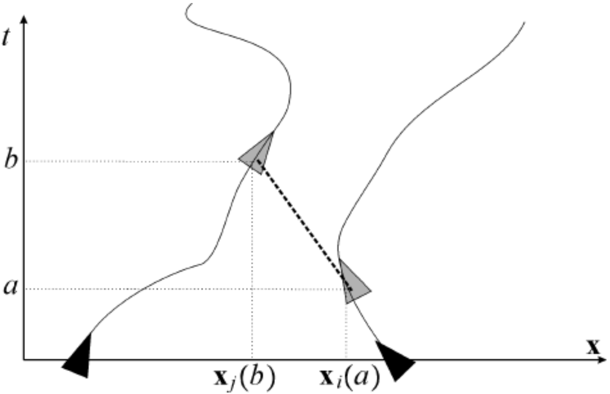
$$\left(x_1(t) - x_1(t')\right)^2 + \left(x_2(t) - x_2(t')\right)^2 - c\left(t - t'\right)^2 = 0.$$

Swarm localization

Consider n robots $\mathscr{R}_1,\dots,\mathscr{R}_n$ described by

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i,\mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i].$$

Omnidirectional sounds are emitted and received.
A *ping* is a 4-uple (a,b,i,j) where a is the emission time, b is the reception time, i is the emitting robot and j the receiver.


t
b
a
x
$\mathbf{x}_j(b)$
$\mathbf{x}_i(a)$

With the time space constraint

$$\begin{array}{l} \dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\ g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \end{array}$$

where

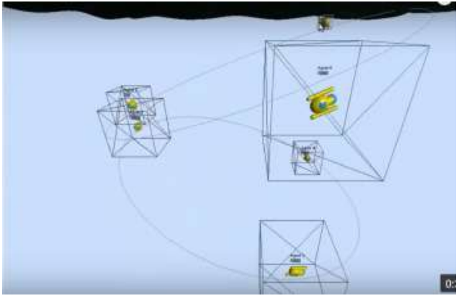
$$g(\mathbf{x}_i, \mathbf{x}_j, a, b) = \|x_1 - x_2\| - c(b - a).$$

Clocks are uncertain. We only have measurements $\tilde{a}(k), \tilde{b}(k)$ of $a(k), b(k)$ thanks to clocks h_i. Thus

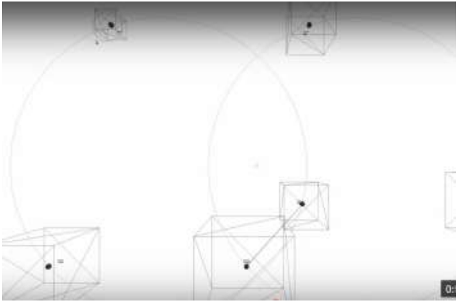
$$\begin{array}{l}
\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}(a(k)) \\
\tilde{b}(k) = h_{j(k)}(b(k))
\end{array}$$

The drift of the clocks is bounded

$$\begin{array}{l}
\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i), \mathbf{u}_i \in [\mathbf{u}_i]. \\
g\left(\mathbf{x}_{i(k)}(a(k)), \mathbf{x}_{j(k)}(b(k)), a(k), b(k)\right) = 0 \\
\tilde{a}(k) = h_{i(k)}(a(k)) \\
\tilde{b}(k) = h_{j(k)}(b(k)) \\
\dot{h}_i = 1 + n_h,\ n_h \in [n_h]
\end{array}$$



https://youtu.be/j-ERcoXF1Ks



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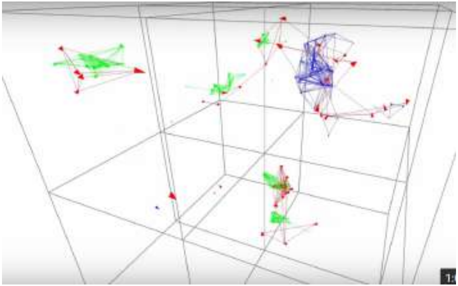

https://youtu.be/GycJxGFvYE8

References

1. Interval analysis [8, 5, 6]
2. Localization with intervals : [7]
3. SLAM with intervals : [4]
4. Interval tubes [9], [3], [1][10]
5. Swarm localization [2]

F. L. Bars, J. Sliwka, O. Reynet, and L. Jaulin.
State estimation with fleeting data.
Automatica, 48(2):381–387, 2012.

A. Bethencourt and L. Jaulin.
Cooperative localization of underwater robots with unsynchronized clocks.
Journal of Behavioral Robotics, 4(4):233–244, 2013.

A. Bethencourt and L. Jaulin.
Solving non-linear constraint satisfaction problems involving time-dependant functions.
Mathematics in Computer Science, 8(3), 2014.

L. Jaulin.
Range-only SLAM with occupancy maps; A set-membership approach.
IEEE Transaction on Robotics, 27(5):1004–1010, 2011.

L. Jaulin, M. Kieffer, O. Didrit, and E. Walter.
Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control and Robotics.
Springer-Verlag, London, 2001.

L. Jaulin, O. Reynet, B. Desrochers, S. Rohou, and J. Ninin.
IaMOOC, Interval analysis with applications to parameter estimation and robot localization ,
www.ensta-bretagne.fr/iamooc/.
ENSTA-Bretagne, 2019.

M. Kieffer, L. Jaulin, E. Walter, and D. Meizel.
Robust autonomous robot localization using interval analysis.
Reliable Computing, 6(3):337–362, 2000.

R. Moore.
Methods and Applications of Interval Analysis.
Society for Industrial and Applied Mathematics, jan 1979.

S. Rohou, L. Jaulin, M. Mihaylova, F. L. Bars, and S. Veres.
Guaranteed Computation of Robots Trajectories.
Robotics and Autonomous Systems, 93:76–84, 2017.

J. A. D. Sandretto and A. Chapoutot.
Dynibex: a differential constraint library for studying dynamical systems.
In *Conference on Hybrid Systems: Computation and Control*, Vienne, Austria, 2016.